QCD Pomeron with conformal spin from AdS/CFT Quantum Spectral Curve

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Motivation

- Using the methods of the recently proposed Quantum Spectral Curve (QSC) originating from integrability of $\mathcal{N} = 4$ Super–Yang-Mills theory analytically continue the scaling dimensions of twist-2 operators and reproduce the so-called pomeron eigenvalue of the Balitsky-Fadin-Kuraev-Lipatov (BFKL) equation with nonzero conformal spin.

- Derive the Faddeev-Korchemsky Baxter equation for the Lipatov's spin chain known from the integrability of the gauge theory in the BFKL limit.

- Find a way for a systematic expansion in the scaling parameter of the BFKL regime.

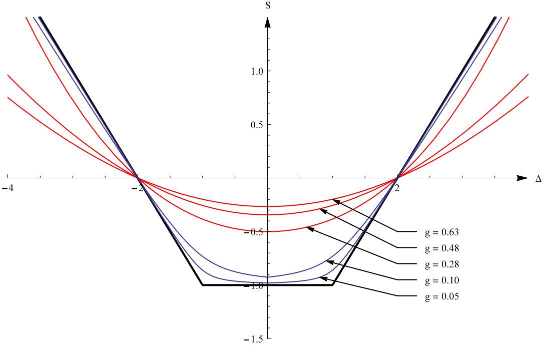
BFKL regime and twist-2 operators in the $\mathcal{N} = 4$ SYM

- We consider important class of operators

$$\mathrm{tr} Z(D_+)^S Z + \text{permutations}$$

For the case with nonzero conformal spin there are also derivatives in the orthogonal directions.

- BFKL scaling is determined by: $S \to -1$, $g \to 0$ and $\frac{g^2}{S+1}$ is finite. Leading order BFKL approximation corresponds to resumming all the powers $\left(\frac{g^2}{S+1}\right)^n$.
- Regge trajectories $S(\Delta)$ corresponding to the twist-2 operator tr $Z(D_+)^S Z$ and different values of g



Generalities of the QSC

- The QSC gives the generalization of the Baxter equation describing the 1-loop spectrum of twist-2 operators to all loops. The spectrum of the $\mathcal{N}=4$ SYM can be described by 16 basic Q-functions, which we denote by $\mathbf{P}_a$, $\mathbf{P}^a$, $\mathbf{Q}_j$ and $\mathbf{Q}^j$, where $a, j = 1, \dots, 4$.
- The AdS/CFT Q-system is formed by 2^8 Q-functions which we denote as $Q_{A|J}(u)$ where $A, J \subset \{1,2,3,4\}$ are two ordered subsets of indices. They satisfy the QQ-relations

$$\begin{aligned} Q_{A|I}Q_{Aab|I} &= Q^+_{Aa|I}Q^-_{Ab|I} - Q^-_{Aa|I}Q^+_{Ab|I}\,,\\ Q_{A|I}Q_{A|Iij} &= Q^+_{A|Ii}Q^-_{A|Ij} - Q^-_{A|Ii}Q^+_{A|Ij}\,,\\ Q_{Aa|I}Q_{A|Ii} &= Q^+_{Aa|Ii}Q^-_{A|I} - Q^+_{A|I}Q^-_{Aa|Ii} \end{aligned}$$

  and reshuffling a pair of individual indices (small letters a, b, i, j) we can express all Q-functions through 8 basic ones. In addition we also impose the constraints $Q_{\emptyset|\emptyset} = Q_{1234|1234} = 1$.
- Another effect which happens at finite coupling is that the poles of Q-functions in the lower-half plane, described above, resolve into cuts $[-2g, 2g]$ (where $g = \sqrt{\lambda}/4\pi$). Finally, we have to introduce new objects – the monodromies μ_{ab} and ω_{ij} corresponding to the analytic continuation of the functions $\mathbf{P}_a$ and $\mathbf{Q}_j$ under these cuts.

Generalities of the QSC

- Here we present our new result which allows for the direct transition between two equivalent systems. As a consequence of the $\mathrm{Q}\,\mathrm{Q}$-relations, $\mathbf{P}$'s and $\mathbf{Q}$'s are related through the following 4th order finite-difference equation

$$0 = \mathbf{Q}^{[+4]}\mathrm{D}_0 - \mathbf{Q}^{[+2]}\left[\mathrm{D}_1 - \mathbf{P}_a^{[+2]}\mathbf{P}^{a\,[+4]}\mathrm{D}_0\right] + \frac{1}{2}\mathbf{Q}\left[\mathrm{D}_2 + \mathbf{P}_a\mathbf{P}^{a\,[+4]}\mathrm{D}_0 + \mathbf{P}_a\mathbf{P}^{a\,[+2]}\mathrm{D}_1\right] + \text{c.c.}$$

where

$$\mathrm{D}_0 = \det\begin{pmatrix} \mathbf{P}^{1[+2]} & \dots & \mathbf{P}^{4[+2]} \\ \mathbf{P}^{1} & \dots & \mathbf{P}^{4} \\ \mathbf{P}^{1[-2]} & \dots & \mathbf{P}^{4[-2]} \\ \mathbf{P}^{1[-4]} & \dots & \mathbf{P}^{4[-4]} \end{pmatrix}, \quad \mathrm{D}_1 = \det\begin{pmatrix} \mathbf{P}^{1[+4]} & \dots & \mathbf{P}^{4[+4]} \\ \mathbf{P}^{1} & \dots & \mathbf{P}^{4} \\ \mathbf{P}^{1[-2]} & \dots & \mathbf{P}^{4[-2]} \\ \mathbf{P}^{1[-4]} & \dots & \mathbf{P}^{4[-4]} \end{pmatrix},$$

$$\mathrm{D}_2 = \det\begin{pmatrix} \mathbf{P}^{1[+4]} & \dots & \mathbf{P}^{4[+4]} \\ \mathbf{P}^{1[+2]} & \dots & \mathbf{P}^{4[+2]} \\ \mathbf{P}^{1[-2]} & \dots & \mathbf{P}^{4[-2]} \\ \mathbf{P}^{1[-4]} & \dots & \mathbf{P}^{4[-4]} \end{pmatrix}.$$

The four solutions of this equation give four functions $\mathbf{Q}_j$.

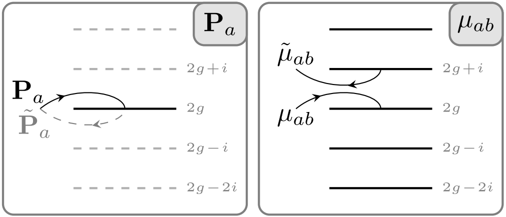
$\mathbf{P}\mu$-system

- We can focus on a much smaller closed subsystem constituted of 8 functions $\mathbf{P}_a$ and $\mathbf{P}^a$, having only one short cut on the real axis on their defining sheet

$$\tilde{\mathbf{P}}_a = \mu_{ab}(u)\mathbf{P}^b \ \ , \ \ \tilde{\mathbf{P}}^a = \mu^{ab}(u)\mathbf{P}_b,$$

  and $\mathbf{P}$'s satisfy the orthogonality relations $\mathbf{P}_a\mathbf{P}^a = 0$.
- The analytic continuation for the μ-functions is given by

$$\tilde{\mu}_{ab}(u) = \mu_{ab}(u+i).$$



- The other equations make the $\mathbf{P}\mu$-system closed

$$\tilde{\mu}_{ab} - \mu_{ab} = \mathbf{P}_a\tilde{\mathbf{P}}_b - \mathbf{P}_b\tilde{\mathbf{P}}_a\ .$$

$\mathbf{Q}\omega$-system

- Knowing $\mathbf{P}_a$ and $\mathbf{Q}_i$ we construct $Q_{a|i}$ which allows us to define ω_{ij}

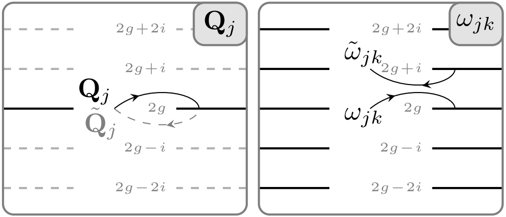
$$\omega_{ij} = Q^-_{a|i} Q^-_{b|j} \mu^{ab} \,.$$

- One can show that $\mathbf{Q}_a$ defined in this way will have one long cut. Also $\hat{\omega}_{ij}$, with short cuts, happens to be periodic $\hat{\omega}^+_{ij} = \hat{\omega}^-_{ij}$, similarly to its counterpart with long cuts $\check{\mu}_{ab}$! Finally, their discontinuities are given by

$$\tilde{\omega}_{ij} - \omega_{ij} = \mathbf{Q}_i \tilde{\mathbf{Q}}_j - \mathbf{Q}_j \tilde{\mathbf{Q}}_i$$

$$\tilde{\mathbf{Q}}_i = \omega_{ij} \mathbf{Q}^j \,.$$

and $\mathbf{Q}$'s satisfy the orthogonality relations $\mathbf{Q}_j \mathbf{Q}^j = 0$.



Asymptotics of $\mathbf{P}$ and $\mathbf{Q}$-functions and their relation to global S^5 and AdS_5 charges

$$\begin{pmatrix} \mathbf{P}_1 \\ \mathbf{P}_2 \\ \mathbf{P}_3 \\ \mathbf{P}_4 \end{pmatrix} \simeq \begin{pmatrix} A_1\, u^{\frac{-J_1-J_2+J_3-2}{2}} \\ A_2\, u^{\frac{-J_1+J_2-J_3}{2}} \\ A_3\, u^{\frac{+J_1-J_2-J_3-2}{2}} \\ A_4\, u^{\frac{+J_1+J_2+J_3}{2}} \end{pmatrix} \quad \begin{pmatrix} \mathbf{P}^1 \\ \mathbf{P}^2 \\ \mathbf{P}^3 \\ \mathbf{P}^4 \end{pmatrix} \simeq \begin{pmatrix} A^1\, u^{\frac{+J_1+J_2-J_3}{2}} \\ A^2\, u^{\frac{+J_1-J_2+J_3-2}{2}} \\ A^3\, u^{\frac{-J_1+J_2+J_3}{2}} \\ A^4\, u^{\frac{-J_1-J_2-J_3-2}{2}} \end{pmatrix}$$

$$\begin{pmatrix} \mathbf{Q}_1 \\ \mathbf{Q}_2 \\ \mathbf{Q}_3 \\ \mathbf{Q}_4 \end{pmatrix} \simeq \begin{pmatrix} B_1\, u^{\frac{+\Delta-S_1-S_2}{2}} \\ B_2\, u^{\frac{+\Delta+S_1+S_2-2}{2}} \\ B_3\, u^{\frac{-\Delta-S_1+S_2}{2}} \\ B_4\, u^{\frac{-\Delta+S_1-S_2-2}{2}} \end{pmatrix} \quad \begin{pmatrix} \mathbf{Q}^1 \\ \mathbf{Q}^2 \\ \mathbf{Q}^3 \\ \mathbf{Q}^4 \end{pmatrix} \simeq \begin{pmatrix} B^1\, u^{\frac{-\Delta+S_1+S_2-2}{2}} \\ B^2\, u^{\frac{-\Delta-S_1-S_2}{2}} \\ B^3\, u^{\frac{+\Delta+S_1-S_2-2}{2}} \\ B^4\, u^{\frac{+\Delta-S_1+S_2}{2}} \end{pmatrix}$$

$$\begin{aligned}
A_1A^1 &= \frac{\left((J_1+J_2-J_3-S_2+1)^2-(\Delta+S_1-1)^2\right)\left((J_1+J_2-J_3+S_2+1)^2-(\Delta-S_1+1)^2\right)}{-16i\,(J_1+J_2+1)\,(J_1-J_3)\,(J_2-J_3+1)} \\
A_2A^2 &= \frac{\left((J_1-J_2+J_3-S_2-1)^2-(\Delta+S_1-1)^2\right)\left((J_1-J_2+J_3+S_2-1)^2-(\Delta-S_1+1)^2\right)}{+16i\,(J_1-J_2-1)\,(J_1+J_3)\,(J_2-J_3+1)} \\
A_3A^3 &= \frac{\left((J_1-J_2-J_3+S_2-1)^2-(\Delta+S_1-1)^2\right)\left((J_1-J_2-J_3-S_2-1)^2-(\Delta-S_1+1)^2\right)}{-16i\,(J_1-J_2-1)\,(J_1-J_3)\,(J_2+J_3+1)} \\
A_4A^4 &= \frac{\left((J_1+J_2+J_3-S_2+1)^2-(\Delta-S_1+1)^2\right)\left((J_1+J_2+J_3+S_2+1)^2-(\Delta+S_1-1)^2\right)}{+16i\,(J_1+J_2+1)\,(J_1+J_3)\,(J_2+J_3+1)}
\end{aligned}$$

QSC for twist-2 operators

- For the twist-2 operators in question, the charges are fixed to $J_2 = J_3 = S_2 = 0$ and $J_1 = 2$, and we will use the notation $S_1 \equiv S \equiv -1 + w$. These operators belong to the so called left-right symmetric sector for which we have the following reduction

$$\mathbf{P}^a = \chi^{ac}\mathbf{P}_c, \qquad \mathbf{Q}^i = \chi^{ij}\mathbf{Q}_j,$$

- The asymptotics are simplified to

$$\begin{aligned}\mathbf{P}_a &\simeq (A_1 u^{-2}, A_2 u^{-1}, A_3, A_4 u)_a,\\ \mathbf{Q}_j &\simeq (B_1 u^{\frac{\Delta+1-w}{2}}, B_2 u^{\frac{\Delta-3+w}{2}}, B_3 u^{\frac{-\Delta+1-w}{2}}, B_4 u^{\frac{-\Delta-3+w}{2}})_j\end{aligned}$$

and

$$\begin{aligned}A_1 A_4 = -A_1 A^1 &= \frac{1}{96i}((5-w)^2 - \Delta^2)((1+w)^2 - \Delta^2),\\ A_2 A_3 = +A_2 A^2 &= \frac{1}{32i}((1-w)^2 - \Delta^2)((3-w)^2 - \Delta^2).\end{aligned}$$

- Prescription for analytic continuation in S. In order to analytically continue the QSC to non-physical domain of non-integer S one should relax the power-like behavior of μ_{ab} (required for all physical states) allowing for the following leading and subleading terms in the asymptotics

$$\mu_{12} \sim \text{const } u^{+\Delta-2} + e^{2\pi u}\text{const } u^{-1-S} + \ldots$$

Leading Order and Next-to-leading Order solutions of the $\mathbf{P}\mu$-system

- After some demanding calculations we get the result for the $\mathbf{P}$-functions

$$\begin{aligned}
\mathbf{P}_1 &\simeq \frac{1}{u} + \frac{2\Lambda w}{u^3}, \\
\mathbf{P}_2 &\simeq \frac{1}{u^2} + \frac{2\Lambda w}{u^4}, \\
\mathbf{P}_3 &\simeq A_3^{(0)} u - \frac{i(\Delta^2-1)^2}{96u} + \left(A_3^{(1)} u + \frac{c_{3,1}^{(2)}}{u\Lambda} - \frac{i(\Delta^2-1)^2\Lambda}{48u^3} \right) w, \\
\mathbf{P}_4 &\simeq A_4^{(0)} + A_4^{(1)} w.
\end{aligned}$$

where $\Lambda = \frac{g^2}{w}$ and

$$c_{3,1}^{(2)} = -\frac{i\Lambda}{24}(\Delta^2-1)\left[2(\Delta^2-1)\Lambda - 1\right].$$

Passing to $\mathbf{Q}\omega$-system

- Substituting the obtained LO $\mathbf{P}$-functions into the 4-th order equation for $\mathbf{Q}$-functions we get a very nice factorization in the LO

$$\left[(u+2i)^2 D + (u-2i)^2 D^{-1} - 2u^2 - \frac{17-\Delta^2}{4}\right]\left[D + D^{-1} - 2 - \frac{1-\Delta^2}{4u^2}\right]\mathbf{Q} = 0,$$

where $D = e^{i\partial_u}$ is the shift operator.

- Thus, we get the equation for $\mathbf{Q}_1$ and $\mathbf{Q}_3$ in the LO

$$\mathbf{Q}_j \frac{\Delta^2 - 1 - 8u^2}{4u^2} + \mathbf{Q}_j^{--} + \mathbf{Q}_j^{++} = 0,$$

which coincides with the Faddeev-Korchemsky Baxter equation for the Lipatov's spin chain after a redefinition $Q = \frac{\mathbf{Q}_j}{u^2}$. It has the following solutions

$$\mathbf{Q}_1(u) = 2iu\, {}_3F_2\left(iu+1, \frac{1}{2} - \frac{\Delta}{2}, \frac{\Delta}{2} + \frac{1}{2}; 1, 2; 1\right),$$

$$\mathbf{Q}_3(u) = \mathbf{Q}_1(-u)\sec\frac{\pi\Delta}{2} + \mathbf{Q}_1(u)\left[-i\coth(\pi u) + \tan\frac{\pi\Delta}{2}\right].$$

- In the NLO the 4-th order equation also factorizes and we obtain the following 2nd order Baxter equation

$$\mathbf{Q}_j\left(\frac{\Delta^2 - 1 - 8u^2}{4u^2} + w\frac{(\Delta^2-1)\Lambda - u^2}{2u^4}\right) + $$
$$+ \mathbf{Q}_j^{--}\left(1 - \frac{iw/2}{u-i}\right) + \mathbf{Q}_j^{++}\left(1 + \frac{iw/2}{u+i}\right) = 0, \qquad j = 1, 3.$$

Calculation of the LO BFKL dimension

- From the NLO 2nd order Baxter equation for $\mathbf{Q}_1$ and $\mathbf{Q}_3$ one can note the following relation between these functions in the LO and NLO

$$\frac{\mathbf{Q}_j^{(1)}(u)}{\mathbf{Q}_j^{(0)}(u)} = +\frac{iw}{2u} + \mathcal{O}(u^0) \ , \ \ j = 1, 3 \, .$$

The key idea of finding the BFKL dimension is to obtain this ratio independently.

- On the other hand we can use the trick

$$\mathbf{Q}_3 = \frac{\mathbf{Q}_3 - \tilde{\mathbf{Q}}_3}{2\sqrt{u^2-4g^2}}\sqrt{u^2-4g^2} + \frac{\mathbf{Q}_3 + \tilde{\mathbf{Q}}_3}{2} =$$
$$= \left[\frac{\mathbf{Q}_3 - \tilde{\mathbf{Q}}_3}{\sqrt{u^2-4g^2}}\right]\left(-\frac{\Lambda w}{u} - \frac{\Lambda^2 w^2}{u^3} + \dots\right) + \text{regular},$$

from where we conclude that we need to express $\tilde{\mathbf{Q}}_3(u)$ in the LO in terms of $\mathbf{Q}_1(u)$ and $\mathbf{Q}_3(u)$.

- It can be done with some effort, which requires to find ω-functions in the first nonvanishing order. This calculation gives the result

$$\tilde{\mathbf{Q}}_1(u) = -\mathbf{Q}_1(-u),$$
$$\tilde{\mathbf{Q}}_3(u) = +\mathbf{Q}_3(-u) - 2\tan\left(\frac{\pi\Delta}{2}\right)\mathbf{Q}_1(-u).$$

Calculation of the LO BFKL dimension

- Combining the previously obtained results, we get

$$\mathbf{Q}_3(u) = -\frac{2i\mathbf{Q}_3^{(0)}(0)\Psi(\Delta)\Lambda w}{u} + \text{regular} + \mathcal{O}(w^2),$$

where

$$\Psi(\Delta) \equiv \psi\left(\frac{1}{2} - \frac{\Delta}{2}\right) + \psi\left(\frac{1}{2} + \frac{\Delta}{2}\right) - 2\psi(1)\ .$$

- Thus, comparing two independent results, we obtain the relation

$$-4\Psi(\Delta)\Lambda = 1\ ,$$

which gives exactly the well-known LO BFKL dimension

$$\frac{1}{4\Lambda} = -\psi\left(\frac{1}{2} - \frac{\Delta}{2}\right) - \psi\left(\frac{1}{2} + \frac{\Delta}{2}\right) + 2\psi(1) + \mathcal{O}(g^2).$$

Conclusions and outlook

- In our work we managed to reproduce the dimension of twist-2 operator of $\mathcal{N} = 4$ SYM theory in the 't Hooft limit in the leading order (LO) of the BFKL regime directly from exact equations for the spectrum of local operators called the Quantum Spectral Curve.

- This is one of a very few examples of all-loop calculations, with all wrapping corrections included, where the integrability result can be checked by direct Feynman graph summation of the original BFKL approach.

- The ultimate goal of the BFKL approximation to QSC would be to find an algorithmic way of generation of any BFKL correction (NNLO, NNNLO, etc) on Mathematica program, similarly to the one for the weak coupling expansion via QSC.

Thanks for your attention!