Solutions of the traveling wave type for Korteweg-de Vries-type system with polynomial potential

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Overview

Introduction

In the theory of plastic deformation, the following infinite-dimensional dynamical system is studied

$$m\ddot{y}_i = y_{i+1} - 2y_i + y_{i-1} + \phi(y_i), \quad i \in \mathbb{Z}, \quad y_i \in \mathbb{R}, \quad t \in \mathbb{R}, \qquad (1)$$

where potential $\phi(\cdot)$ is given by a smooth periodic function. The equation (1) is a system with the Frenkel-Kontorova potential [Frenkel & Contorova, 1938]. Such a system is a finite difference analog of the nonlinear wave equation. It simulates the behavior of a countable number of balls of mass m placed at integer points of the numerical line, where each pair of adjacent balls is connected by an elastic spring, and describes the propagation of longitudinal waves in an infinite homogeneous absolutely elastic rod. The study of such systems with different potentials is one of the intensively developing directions in the theory of dynamical systems. For these systems, the central task is to study solutions of the traveling wave type as one of the observed wave classes.

Introduction

Definition

We say that the solution $\{y_i(\cdot)\}_{-\infty}^{+\infty}$ of the system (1), defined for all $t \in \mathbb{R}$, has a traveling wave type, if there is $\tau > 0$, independent of t and i, that for all $i \in \mathbb{Z}$ and $t \in \mathbb{R}$ the following equality holds

$$y_i(t+\tau) = y_{i+1}(t).$$

The constant τ will be called a *characteristic* of the traveling wave.

Introduction

The proposed approach is based on the existence of a one-to-one correspondence of solutions of the traveling wave type for infinite-dimensional dynamical systems with solutions of induced FDEPT [Beklaryan, 2007]. To study the existence and uniqueness of solutions of the traveling wave type, it is proposed to localize solutions of induced FDEPT in spaces of functions, majorized by functions of a given exponential growth. This approach is particularly successful for systems with Frenkel-Kontorova potentials. In this way, it is possible to obtain a "correct" extension of the concept of a traveling wave in the form of solutions of the quasi-traveling wave type, which is related to the description of processes in inhomogeneous environments for which the set of traveling wave solutions is trivial [Beklaryan, 2010, 2014].

For the infinite-dimensional dynamical system under consideration, the study of solutions of the traveling wave type with the characteristic τ, i.e. solutions of the system

$$\ddot{y}_i = m^{-1}(y_{i+1} - 2y_i + y_{i-1} + \phi(y_i)), \quad i \in \mathbb{Z}, \quad t \in \mathbb{R},$$
$$y_i(t+\tau) = y_{i+1}(t)$$

turns out to be equivalent to the study of a solution space of the induced FDEPT

$$\ddot{x}(t) = m^{-1}(x(t+\tau) - 2x(t) + x(t-\tau) + \phi(x(t))), \quad t \in \mathbb{R}.$$

In this case, the corresponding solutions are related as follows: for any $t \in \mathbb{R}$

$$x(t) = y_{[t\tau^{-1}]}(t - \tau[t\tau^{-1}]),$$

where $[\cdot]$ means the integer part of a number. And vice versa, the corresponding solution of the traveling wave type is determined by the rule

$$y_0(t) = x(t), \quad y_i(t) = y_0(t+i\tau), \quad i \in \mathbb{Z}, \quad t \in \mathbb{R}.$$

Functional Differential Equations

The theory of the functional-differential equations developed in the works of many authors, among which it is necessary to single out the works of Myshkis, Bellman, Kato, Krasovsky, Krasnoselsky, Sharkovsky, Hale, Varga, and others.

We consider a functional-differential equation of pointwise type (FDEPT)

$$\dot{x}(t) = f(t, x(q_1(t), \ldots, x(q_s(t))), \quad t \in B_R, \tag{2}$$

where $f : \mathbb{R} \times \mathbb{R}^{ns} \longrightarrow \mathbb{R}^n$ is a mapping of the $C^{(0)}$ class; $q_j(\cdot)$, $j = 1, \ldots, s$ are homeomorphisms of the real line preserving orientation; B_R is either closed interval $[t_0, t_1]$ or closed half-line $[t_0, +\infty[$ or the real line $\mathbb{R}$.

Functional Differential Equations

Definition

An absolutely continuous function $x(\cdot)$ defined on $\mathbb{R}$ is called a solution of the equation (2) if for almost all $t \in B_R$ the function $x(\cdot)$ satisfies this equation. If, in addition, $x(\cdot) \in C^{(k)}(\mathbb{R}, \mathbb{R}^n), k = 0, 1, \ldots$ then this solution is called a solution of the class $C^{(k)}$.

Functional Differential Equations

FDEPT under consideration:

- is an ordinary differential equation if $q_j(t) \equiv t, j = 1, \dots, s$;
- is an equation with pure delays if $q_j(t) \leq t, j = 1, \dots, s$;
- is an equation with pure advances if $q_j(t) \geq t, j = 1, \dots, s$.

The functions $[q_j(t) - t], j = 1, \dots, s$ are called deviations of the argument. Using time replacement, we can always achieve the condition

$$h = \max_{i \in \{1,\dots,s\}} h_j < +\infty, \quad h_j = \sup_{t \in \mathbb{R}} |q_j(t) - t|, \quad j = 1, \dots, s$$

for deviations of the argument. It is obvious that such a replacement of time can change the growth character of the right-hand side of the equation with respect to the time variable.

Functional Differential Equations

The approach proposed for the study of such equations is based on a formalism whose central element is the construction using a finitely generated group

$$Q =< q_1, \dots, q_s >$$

of homeomorphisms of the line. The considered type of FDEPT is rather wide and, in particular, describes traveling wave solutions (soliton solutions) for finite difference analogs of the equations of mathematical physics. At the same time, the use of the specifics of such a class of equations associated with group features allows us to obtain advanced results for them.

Functional Differential Equations

The essence of the approach is that the infinite-dimensional vector-function

$$\{x(q(t))\}_{q\in Q}$$

constructed by a solution $x(\cdot)$ of the equation (2) will be the solution of some induced infinite-dimensional ordinary differential equation. Let's consider the full direct product

$$\mathcal{K}^n_Q = \overline{\prod}_{q\in Q}\mathbb{R}^n_q, \quad \mathbb{R}^n_q = \mathbb{R}^n, \quad \varkappa \in \mathcal{K}^n_Q, \quad \varkappa = \{x_q\}_{q\in Q}.$$

The translation group $\mathbb{T}_Q =< T_{q_1}, \ldots, T_{q_s} >$ of the phase space $\mathcal{K}^n_Q$ is defined by the rule: for any $\bar{q} \in Q$

$$T_{\bar{q}}\{x_q\}_{q\in Q} = \{x_{q\bar{q}}\}_{q\in Q}.$$

Let's consider the case when $B_R = \mathbb{R}$ and the group Q is a group of diffeomorphisms. In this case the right-hand side of the equation (2) induces a map

$$F : \mathbb{R} \times \mathcal{K}_Q^n \to \mathcal{K}_Q^n,$$

which satisfies to the permutable relation: for any $t \in \mathbb{R}, \varkappa \in \mathcal{K}_Q^n, q \in Q$

$$T_q F(t, \varkappa) = \dot{q}(t) F(q(t), T_q \varkappa)$$

Studying of solutions of the equation (2) is equivalent to studying of solutions of the infinite-dimensional ordinary differential equation satisfying to group of nonlocal restrictions

$$\dot{\varkappa}(t) = F(t, \varkappa), \quad t \in \mathbb{R}, \tag{3}$$

$$\varkappa(q(t)) = T_q \varkappa(t), \quad t \in \mathbb{R}, \quad q \in Q. \tag{4}$$

Restrictions (4) mean that any shift on time along the solution corresponds to some shift on space of the solution.

Initial-boundary value problem

The main goal in the study of such differential equations is the investigation of the initial-boundary value problem

$$\dot{x}(t) = f(t, x(q_1(t), \ldots, x(q_s(t))), \quad t \in B_R, \tag{5}$$

$$\dot{x}(t) = \varphi(t), \quad t \in \mathbb{R} \backslash B_R, \quad \varphi(\cdot) \in L_\infty(\mathbb{R}, \mathbb{R}^n), \tag{6}$$

$$x(\bar{t}) = \bar{x}, \quad \bar{t} \in \mathbb{R}, \quad \bar{x} \in \mathbb{R}^n, \tag{7}$$

which we will call the *basic initial-boundary value problem*. In a general situation, when $\bar{t} \neq t_0, t_1$, or deviations of the argument are arbitrary, we have a problem with *non-local initial-boundary conditions*.

Initial-boundary value problem

If there is no deviation of the argument $(q_j(t) \equiv t, j = 1, \ldots, s)$, the boundary value problem becomes the Cauchy problem for the ordinary differential equation.

The solution of the equation (5) is any absolutely continuous function $x(t)$, $t \in \mathbb{R}$ that satisfies this equation almost everywhere. *The solution of the basic initial-boundary value problem* is any solution of the equation (5) that satisfies the boundary condition (6) and the initial condition (7). If $B_R = [t_0, t_1]$ or $B_R = [t_0, +\infty[$, then for $\bar{t} = t_0$ and for an equation with delays $(q_j(t) \leq t, j = 1, \ldots, s)$, or for $\bar{t} = t_1$ and for an equation with advances $(q_j(t) \geq t, j = 1, \ldots, s)$, the boundary value problem is transformed into the well-known formulation of the *initial problem* for an equation with delays or advances in the argument. It is important that in the noted cases, the problem under consideration has *initial-boundary conditions of local type*.

Main results

Let's define a Banach space of functions $x(\cdot)$ with weights

$$\mathcal{L}_\mu^n C^{(k)}(\mathbb{R}) = \Big\{ x(\cdot) : x(\cdot) \in C^{(k)}\left(\mathbb{R}, \mathbb{R}^n\right), \\ \max\nolimits_{0\le r\le k} \sup\nolimits_{t\in\mathbb{R}} \|x^{(r)}(t)\mu^{|t|}\|_{\mathbb{R}^n} < +\infty \Big\}, \quad \mu \in (0,1),$$

with a norm

$$\|x(\cdot)\|_\mu^{(k)} = \max_{0\le r\le k} \sup_{t\in\mathbb{R}} \|x^{(r)}(t)\mu^{|t|}\|_{\mathbb{R}^n}.$$

Main results

Let's formulate a system of restrictions on the right-hand side of FDEPT:

(a) $f(\cdot) \in C^{(0)}(\mathbb{R} \times \mathbb{R}^{n\times s}, \mathbb{R}^n)$ (function $f(\cdot)$ with respect to the variable t can be considered as piecewise continuous function with discontinuities of the first kind at the points of a discrete set);

(b) quasilinear growth condition: for any $t, z_j, \bar{z}_j, j = 1, \dots, s$

$$\|f\left(t, z_1, \dots, z_s\right)\|_{\mathbb{R}^n} \leq M_0(t) + M_1 \sum_{j=1}^{s} \|z_j\|_{\mathbb{R}^n},\ M_0(\cdot) \in C^{(0)}(\mathbb{R}, \mathbb{R})$$

and Lipschitz condition

$$\|f\left(t, z_1, \dots, z_s\right) - f\left(t, \bar{z}_1, \dots, \bar{z}_s\right)\|_{\mathbb{R}^n} \leq M_2 \sum_{j=1}^{s} \|z_j - \bar{z}_j\|_{\mathbb{R}^n}$$

(in fact $M_1 \leq M_2$, but M_1 and M_2 can be taken equal);

Main results

(c) there exists $\mu^* \in \mathbb{R}_+$ such that the expression

$$\sup_{i\in\mathbb{Z}} M_0(t+i)\,(\mu^*)^{|i|}$$

has a finite value for any $t \in \mathbb{R}$ and is continuous as a function of the argument t.

(d) for the μ^* from the item (c) the family of functions

$$\tilde{f}_{i,z_1,\ldots,z_s}(t) = f(t+i, z_1, \ldots, z_s)(\mu^*)^{|i|},\ i \in \mathbb{Z},\ z_1, \ldots, z_s \in \mathbb{R}^n$$

is equicontinuous on any finite interval.

Main results

The right-hand side $f(\cdot)$ of FDEPT will be considered as an element of the Banach space $V_{\mu^*}(\mathbb{R} \times \mathbb{R}^{ns}, \mathbb{R}^n)$

$$V_{\mu^*}(\mathbb{R} \times \mathbb{R}^{ns}, \mathbb{R}^n) = \left\{ f(\cdot) : f(\cdot) \text{ satisfies the conditions } (a) - (d) \ \right\},$$
$$\|f(\cdot)\|_{Lip} = \sup_{t \in \mathbb{R}} \|f(t, 0, \ldots, 0)(\mu^*)^{|t|}\|_{\mathbb{R}^n} +$$
$$+ \sup_{(t, z_1, \ldots, z_s, \bar{z}_1, \ldots, \bar{z}_s) \in \mathbb{R}^{1+2ns}} \frac{\|f(t, z_1, \ldots, z_s) - f(t, \bar{z}_1, \ldots, \bar{z}_s)\|_{\mathbb{R}^n}}{\sum_{j=1}^{s} \|z_j - \bar{z}_j\|_{\mathbb{R}^n}},$$

where the parameter $\mu^* \in \mathbb{R}_+$ coincides with the corresponding constant from the condition (c). Obviously, for the function $f(\cdot) \in V_{\mu^*}(\mathbb{R} \times \mathbb{R}^{ns}, \mathbb{R}^n)$ the smallest value of the constant M_2 from the Lipschitz condition (the condition (b)) coincides with the value of the second summand in the definition of the norm $f(\cdot)$. In what follows, speaking of the Lipschitz condition, by the constant M_2 we mean exactly its smallest value.

We have the following theorem on the existence and uniqueness of a solution.

Theorem 1 ([Beklaryan, 1991])

If for some $\mu \in (0, \mu^*) \cap (0, 1)$ *the inequality*

$$M_2 \sum_{j=1}^{s} \mu^{-|h_j|} < \ln \mu^{-1},$$

is satisfied then for any fixed initial-boundary conditions

$$\varphi(\cdot) \in L_\infty(\mathbb{R}, \mathbb{R}^n), \quad \bar{x} \in \mathbb{R}^n$$

there exists a solution (absolutely continuous) $x(\cdot) \in \mathcal{L}_\mu^n C^{(0)}(\mathbb{R})$ *of the basic initial-boundary value problem* (5)–(7). *Such a solution is unique and, as an element of the space* $\mathcal{L}_\mu^n C^{(0)}(\mathbb{R})$, *depends continuously on the initial-boundary conditions* $\varphi(\cdot) \in L_\infty(\mathbb{R}, \mathbb{R}^n), \bar{x} \in \mathbb{R}^n$ *and the right-hand side of the equation (function* $f(\cdot) \in V_{\mu^*}(\mathbb{R} \times \mathbb{R}^{ns}, \mathbb{R}^n)$*).*

Main results

Let's formulate sufficient conditions of a new type for the existence of periodic and bounded solutions for ordinary differential equations, as well as for FDEPT. Such conditions are based on taking into account the asymptotic properties of solutions of differential equations that were not taken into account in the previous study of periodic and bounded solutions. For simplicity, we assume that all deviations are integer, i.e. we consider the equation

$$\dot{x}(t) = f\left(t, x(t+n_1), \ldots, x(t+n_s)\right), \quad t \in \mathbb{R}. \tag{8}$$

Main results

Let's introduce the notation

$$\mathbb{A}_f = \frac{(\mu^{-\omega} - 1)}{\ln \mu^{-1}}, \quad \mathbb{B}_f = \frac{M_2 \sum_{j=1}^{s} \mu^{-|n_j|}}{\left(\ln \mu^{-1} - M_2 \sum_{j=1}^{s} \mu^{-|n_j|}\right)},$$

$$\mathbb{C}_f(r) = \left[M_2 sr + \sup_{t\in\mathbb{R}} \|f(t, 0, \ldots, 0)\|_{\mathbb{R}^n}\right],$$

$$M_{0\infty\mu}(t) = \sup_{i\in\mathbb{Z}} M_0(t+i)\mu^{|i|},$$

$$\tilde{\mathbb{C}}_f(r) = \left[M_2 sr + \inf_{\xi\in[0,\omega]} \sup_{\tau\in[\xi,\xi+1]} M_{0\infty\mu}(\tau)\right].$$

It is not difficult to see that condition $\tilde{\mathbb{C}}_g(r) \leq \mathbb{C}_g(r)$ holds.

Theorem 2 (in publ.)

Suppose that the map $f(\cdot)$ satisfies the conditions $(a)-(d)$ and function $f(t,0,\ldots,0), t\in\mathbb{R}$ is uniformly bounded. If for given $\mu\in(0,\mu^*)\cap(0,1)$, $r>0$ and for all $\bar{x}\in\mathbb{R}^n, \|\bar{x}\|_{\mathbb{R}^n}=r, t\in\mathbb{R}$ it is true that

$$M_2\sum_{j=1}^{s}\mu^{-|n_j|}<\ln\mu^{-1},\quad \sup_{t\in\mathbb{R}}\left(\frac{\bar{x}}{\|\bar{x}\|_{\mathbb{R}^n}},\int_t^{t+\omega}f(\tau,\bar{x},\ldots,\bar{x})d\tau\right)<$$
$$<-\mathbb{A}_f\mathbb{B}_f\mathbb{C}_f(r)-\frac{1}{2r}\left[\mathbb{A}_f(\mathbb{B}_f+1)\mathbb{C}_f(r)\right]^2.$$

then for the initial FDEPT (8) there exists bounded solution $x(\cdot)$, which lies in the ball of the space $\mathbb{R}^n$ with radius $\mu^{-\omega}\hat{\mathcal{R}}$, where

$$\hat{\mathcal{R}}=r+\mathbb{C}_f(r)\,[\ln\mu^{-1}-M_2\sum_{j=1}^{s}\mu^{-|n_j|}]^{-1}.$$

Moreover, the length of the maximum open intervals of the set $\mathbb{P}=\mathbb{R}\setminus\{t: t\in\mathbb{R}, \|x(t)\|_{\mathbb{R}^n}\le r\}$ is less than ω.

Main results

On the basis of Theorem 1 we are going to formulate a proposition on the approximation of solutions of an initial-boundary value problem defined on the whole line by solutions of the initial-boundary value problem defined on the interval $[-k, k]$ as $k \to +\infty$. We consider the initial-boundary value problem on the whole line $B_R = \mathbb{R}$

$$\dot{x}(t) = f(t, x(t+n_1), \ldots, x(t+n_s)), \quad t \in \mathbb{R}, \tag{9}$$

$$x(\bar{t}) = \bar{x}, \quad \bar{t} \in \mathbb{R}, \quad \bar{x} \in \mathbb{R}^n \tag{10}$$

and for each $k \in \mathbb{Z}$ the initial-boundary value problem on a finite interval $B_R = [-k, k]$

$$\dot{x}(t) = f(t, x(t+n_1), \ldots, x(t+n_s)), \quad t \in [-k, k], \tag{11}$$

$$\dot{x}(t) = \varphi(t), \quad t \in \mathbb{R}\backslash[-k, k], \quad \varphi(\cdot) \in \mathcal{L}_1^n L_\infty(\mathbb{R}), \tag{12}$$

$$x(\bar{t}) = \bar{x}, \quad \bar{t} \in \mathbb{R}, \quad \bar{x} \in \mathbb{R}^n. \tag{13}$$

Main results

Theorem 3 (Beklaryan & Beklaryan [2019])

If for $\mu \in (0, \mu^\star) \cap (0, 1)$ *the inequality*

$$M_2 \sum_{j=1}^{s} \mu^{-|n_j|} < \ln \mu^{-1}, \tag{14}$$

is satisfied, and (μ_1, μ_2) *is the maximal interval on which the inequality* (14) *holds, then for any* $\bar{x} \in \mathbb{R}^n, \varphi(\cdot) \in \mathcal{L}_1^n L_\infty(\mathbb{R})$ *the solution* $\hat{x}(\cdot)$ *of the initial-boundary value problem* (9)–(10)*, as an element of the space* $\mathcal{L}_\mu^n C^{(0)}(\mathbb{R})$*, is approximated by solutions* $\hat{x}_k(\cdot)$ *of the initial-boundary value problem* (11)–(13) *as* $k \to +\infty$ *and for any arbitrarily small number* ϵ, $0 < \epsilon < \mu_2 - \mu_1$ *there is* $C_{f,\epsilon}$ *such that the following estimate is true*

$$\|\hat{x}(\cdot) - \hat{x}_k(\cdot)\|_\mu^{(0)} \leq C_{f,\epsilon} \left(\frac{\mu_1}{\mu_2 - \epsilon} \right)^k .$$

Korteweg-de Vries equation

There has been a particular interest in the theory of Korteweg-de Vries (KdV) equation due to its significance in nonlinear dispersive wave theory. Many different real-world nonlinear physical problems are modeled by this well-known equation [Gardner et al., 1967; Kortweg & De Vries, 1895]. For example, this equation has many direct physical applications to solids, liquids, gases, pedestrian flow models [Xu et al., 2013], car-following models [Sun et al., 2018; Yu et al., 2014] and so on. It has widely been argued and accepted [Hale & Verduyn Lunel, 1993; Wu, 1996] that for various reasons, time delay should be taken into consideration in modeling. Zhao and Xu [Zhao & Xu, 2010] has considered solitary wave solutions of the KdV equation with delays, also Zhao dealt with the initial-value problem of the delay KdV equation [Zhao et al., 2012]. Therefore, we want to incorporate a single discrete time delay $\tau > 0$ into KdV equation and consider the delay KdV-type equation's initial-boundary value problem.

Korteweg-de Vries equation

KdV-type class of equations has different versions of the representation, but further we will concentrate on finding numerical solutions for the equation of the following form

$$\ddot{x}(t) = c[\alpha x(t) + (1-\alpha)x(t-c\tau)] + \frac{1}{2}x^2(t) + \tau\dot{x}(t-c\tau), \qquad (15)$$

$$t \in \mathbb{R}, \quad \alpha \in [0,1], \quad c,\tau \in \mathbb{R}_+.$$

It should also be noted that following [Zhao & Xu, 2010], the second-order FDEPT (15) can be obtained by integrating an equation after substitution a solitary wave solution $U(y,t) = x(y+ct)$ to the KdV equation with time delay

$$[\alpha U_t(y,t) + (1-\alpha)U_t(y,t-\tau)] + U(y,t)U_y(y,t) +$$
$$\tau U_{yy}(y,t-\tau) - U_{yyy}(y,t) = 0.$$

In the previous sections, there was noted a one-to-one correspondence between solutions of traveling wave type for infinite-dimensional systems of ordinary differential equations and solutions of induced FDEPT. In particular, such infinite-dimensional differential equations can arise as finite-difference analogues of continuous systems. Here we consider traveling waves for the KdV equation as a continuous system. Moreover, we consider KdV equation with a delay. The arising equations, satisfied by traveling waves, are also FDEPT. For such equations with a quasilinear right-hand side, conditions for the existence of bounded solutions were obtained, as well as an estimate of the radius of the ball in which such solutions change. For equations of traveling waves with a polynomial right-hand side, redefining the right-hand side outside a certain sphere, one can achieve that it becomes quasilinear. Applying the corresponding result on the existence of a bounded solution for equations with a quasilinear right-hand side, one can obtain bounded solutions for the initial equation of traveling waves, and also a description of the range of parameters under which this boundedness takes place.

Optimization problem

Let's give a formal statement of the optimization problem for the search for numerical solutions of FDEPT. We consider the system

$$F_i(t, \dot{x}(t), x(t+n_1), \dots, x(t+n_s)) = 0, \quad i = \overline{1,k}, \quad t \in [t_l, t_r],$$

where $F : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{ns} \longrightarrow \mathbb{R}^n$ – mapping of $C^{(0)}$ class; $n_j \in \mathbb{Z}$, $j = \overline{1,s}$; $t_l, t_r \in \mathbb{R}$; the values of the derivatives of the phase variables are defined on the extended interval $t \in [t_{ll}, t_{rr}]$, $t_{ll} = t_l + \min\{0, n_1, \dots, n_s\}$, $t_{rr} = t_r + \max\{0, n_1, \dots, n_s\}$

$$\dot{x}(t) = h_l(t), \quad t \in [t_{ll}, t_l],$$
$$\dot{x}(t) = h_r(t), \quad t \in [t_r, t_{rr}],$$

where $h_l, h_r : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ – mapping of $C^{(0)}$ class. The initial-boundary conditions are given by the functionals

$$K_m(\dot{x}(\tau), x(\tau_1), \dots, x(\tau_p)) = 0, \quad m = \overline{1,q}, \quad \tau, \tau_i \in [t_l, t_r], \quad i = \overline{1,p}.$$

Optimization problem

The optimization problem consists in finding a trajectory $\hat{x}(t)$ that delivers the minimum of the residual functional

$$I(\hat{x}(t)) = v^{(N)} \left(\sum_{i=1}^{k} \int_{t_l}^{t_r} F_i^2(t, \dot{\hat{x}}(t), \hat{x}(t+n_1), \ldots, \hat{x}(t+n_s))dt + \int_{t_{ll}}^{t_l} [\dot{\hat{x}}(t) - h_l(t)]^2 dt + \int_{t_r}^{t_{rr}} [\dot{\hat{x}}(t) - h_r(t)]^2 dt \right) + v^{(K)} \sum_{m=1}^{q} K_m^2(\dot{\hat{x}}(\tau), \hat{x}(\tau_1), \ldots, \hat{x}(\tau_p)),$$

where $v^{(N)}, v^{(K)} \in \mathbb{R}_+$ – weighting coefficients.

Numerical methods

The proposed approach to the investigation of boundary value problems is based on the Ritz method and spline-collocation constructions and was implemented in [Beklaryan & Beklaryan, 2017; Zarodnyuk et al., 2016, 2017]. To solve the problems of the class under consideration, the trajectories of the system are discretized on a grid with a constant step, and a generalized residual functional (RF) is formulated that includes both the weighted residual of the original differential equation and the residual of the boundary conditions. A spline differentiation technique is used, based on two spline approximation designs: using cubic natural splines and using a special type of spline whose second derivatives at the edges are also controlled using optimized parameters. For solving the stated finite-dimensional problems, a set of algorithms for local optimization (quasi-Newtonian method BFGS; two versions of the Powell's method; Barzilai-Borwein method; version of the method of confidence domains, and others) and global optimization (method of random multistart; method of curvilinear searching; tunnel method; parabolic method, and others) was implemented.

Software complex *OPTCON-F*

The corresponding software complex (SC) *OPTCON-F* was implemented in the language *C* under the control of operating systems *OS Windows*, *OS Linux* and *Mac OS* using compilers *BCC 5.5* and *GCC*. SC was designed to obtain a numerical solution of boundary value problems, parametric identification problems and optimal control for dynamical systems described by FDEPT [Gornov et al., 2013].
The heuristic search algorithm for the solution $\hat{x}(t)$ can be justified on the basis of the existence and uniqueness theorems for initial-boundary value problems for the investigated FDEPT, as well as theorems on approximating solutions of such equations on the whole line by solutions of the initial-boundary value problem on a sequence of expanding intervals. A description of such equations and corresponding results was presented in the previous sections.

Software complex *OPTCON-F*

In the SC *OPTCON-F* there is the possibility of sequential application of various algorithms within the framework of constructing a solution for one task. Thus, the constructed intermediate solution in the previous step becomes the starting solution for the following algorithm. In this case, such an implementation does not prevent the global search algorithms from “popping out” of the local solution. Separately, we note the presence of a programming module that allows predetermining the order of application of algorithms, as well as the construction of complex chain of steps depending on the current or historical values of a number parameters. For the examples presented below, the following scheme was used in cycle: the generalized quasi-Newtonian and Powell-Brent’s methods were used alternately, and after the error changed by less than 10^{-h} the adaptive modification of the Hooke-Jeeves method was used l times. The stopping criterion depended on the number of iterations in the first part of the cycle, as well as the current error estimate and its dynamics.

Problem statement

Next, the results of the computational experiments on the study of initial-boundary value problems for systems of FDEPT using OPTCON-F software will be presented. Let's consider FDEPT of the following form

$$\ddot{x}(t) = c[\alpha x(t) + (1-\alpha)x(t-c\tau)] + \frac{1}{2}x^2(t) + \tau\dot{x}(t-c\tau), \quad t \in \mathbb{R}, \tag{16}$$

where $\alpha \in [0,1], c, \tau \in \mathbb{R}_+$. Using a time-variable transformation the equation (16) can be rewritten in the form of the following system of equations of the first order

$$\begin{cases} \dot{z}_1(t) = c\tau z_2(t), \\ \dot{z}_2(t) = c\tau\left(c[\alpha z_1(t) + (1-\alpha)z_1(t-1)] + \frac{1}{2}z_1^2(t) + \tau z_2(t-1)\right). \end{cases} \tag{17}$$

Example 1

We consider dynamical system in the following form:

$$\begin{cases} \dot{z}_1(t) = 0.01 z_2(t), \\ \dot{z}_2(t) = 0.01 \left(0.1 z_1(t) + 0.9 z_1(t-1) + \frac{1}{2} z_1^2(t) + 0.01 z_2(t-1)\right), \end{cases}$$

$t \in \mathbb{R}$,

initial conditions

$$\begin{cases} z_1(0) = -5, \\ z_2(0) = 0. \end{cases} \tag{18}$$

Here, with respect to the system (17), we have $\alpha = 0.1, \tau = 0.01, c = 1$.

Taking into account the impossibility of considering the numerical solution of the system on an infinite interval, we introduce the parameter k and the corresponding family of expanding initial-boundary value problems

$$\begin{cases} \dot{z}_1(t) = 0.01 z_2(t), \\ \dot{z}_2(t) = 0.01\left(0.1 z_1(t) + 0.9 z_1(t-1) + \frac{1}{2} z_1^2(t) + 0.01 z_2(t-1)\right), \end{cases}$$

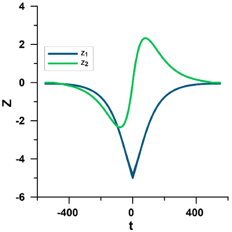
$t \in [-k, k]$,

boundary conditions

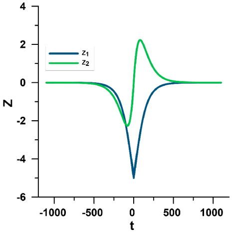
$$\begin{cases} \dot{z}_1(t) = 0, \\ \dot{z}_2(t) = 0, \end{cases} \quad t \in (-\infty, -k] \cup [k, +\infty),$$

initial conditions

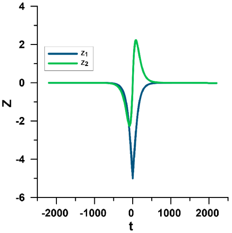
$$\begin{cases} z_1(0) = -5, \\ z_2(0) = 0. \end{cases} \tag{19}$$



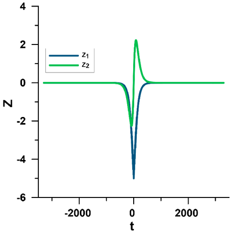
(a) $k = 500,\ RF \simeq 3.1 \times 10^{-5}$



(b) $k = 1000,\ RF \simeq 1.6 \times 10^{-5}$



(c) $k = 2000,\ RF \simeq 7.5 \times 10^{-6}$



(d) $k = 3000,\ RF \simeq 4.7 \times 10^{-6}$

Figure 1: Trajectories of the system (19) at different k.

Example 1

Since the equation (16) is autonomous, the solution space of such equation is invariant with respect to time-variable shifts. Therefore, it suffices to consider a family of solutions of the initial problem (18) with a different values of $z_1(0)$. Figure 2 shows the integral curves for different values of the parameter $r = z_1(0)$ for the system (19). Note that for values of $|r|$ greater than about 20, "destruction" of traveling waves occurs and the residual functional of the numerical solution begins to grow rapidly, although for smaller values of $|r|$ it does not exceed 3×10^{-4}.

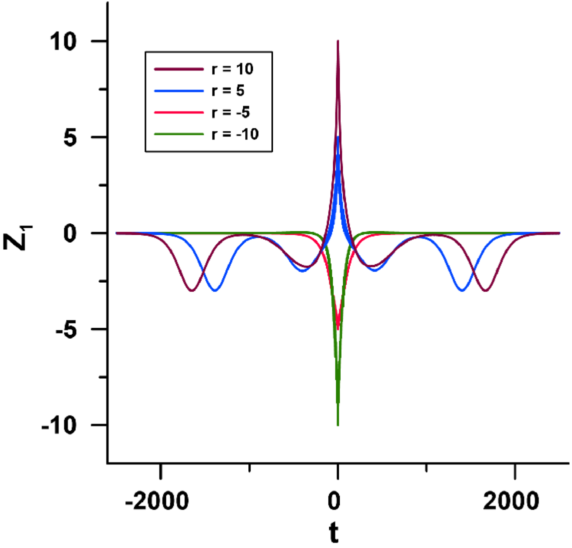


Figure 2: Trajectories of the system (19) at different r

Example 2

Next we consider dynamical system in the following form:

$$\begin{cases} \dot{z}_1(t) = 0.015 z_2(t), \\ \dot{z}_2(t) = 0.015\left(0.45 z_1(t) + 1.05 z_1(t-1) + \dfrac{1}{2} z_1^2(t) + 0.01 z_2(t-1)\right), \end{cases}$$

$t \in \mathbb{R}$,

initial conditions

$$\begin{cases} z_1(0) = -5, \\ z_2(0) = 0. \end{cases} \tag{20}$$

Here, with respect to the system (17), we have
$\alpha = 0.3, \tau = 0.01, c = 1.5.$

Again taking into account the impossibility of considering the numerical solution of the system on an infinite interval, we introduce the parameter k and the corresponding family of expanding initial-boundary value problems

$$\begin{cases} \dot{z}_1(t) = 0.015 z_2(t), \\ \dot{z}_2(t) = 0.015\left(0.45 z_1(t) + 1.05 z_1(t-1) + \frac{1}{2} z_1^2(t) + 0.01 z_2(t-1)\right), \end{cases}$$

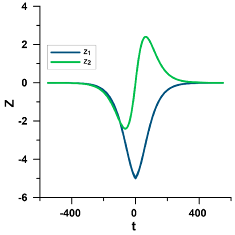
$t \in [-k, k]$,

boundary conditions

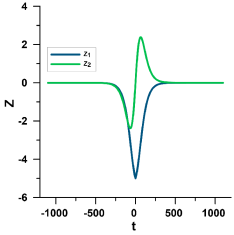
$$\begin{cases} \dot{z}_1(t) = 0, \\ \dot{z}_2(t) = 0, \end{cases} \quad t \in (-\infty, -k] \cup [k, +\infty),$$

initial conditions

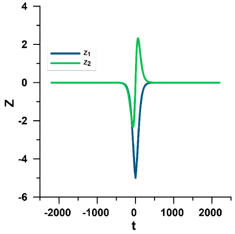
$$\begin{cases} z_1(0) = -5, \\ z_2(0) = 0. \end{cases} \tag{21}$$



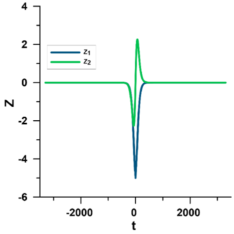
(a) $k = 500,\ RF \simeq 3.3 \times 10^{-6}$



(b) $k = 1000,\ RF \simeq 1.7 \times 10^{-6}$



(c) $k = 2000,\ RF \simeq 9.1 \times 10^{-7}$



(d) $k = 3000,\ RF \simeq 7.2 \times 10^{-7}$

Figure 3: Trajectories of the system (21) at different k.

Example 2

Since the equation (16) is autonomous, the solution space of such equation is invariant with respect to time-variable shifts. Therefore, it suffices to consider a family of solutions of the initial problem (20) with a different values of $z_1(0)$. Figure 4 shows the integral curves for different values of the parameter $r = z_1(0)$ for the system (21). Note that for values of $|r|$ greater than about 10, “destruction” of traveling waves also occurs and the residual functional of the numerical solution begins to grow rapidly, although for smaller values of $|r|$ it does not exceed 4×10^{-5}. Thus, with increasing delay, the solution oscillates on the support around zero, and the threshold of the initial value for the destruction of traveling waves decreases.

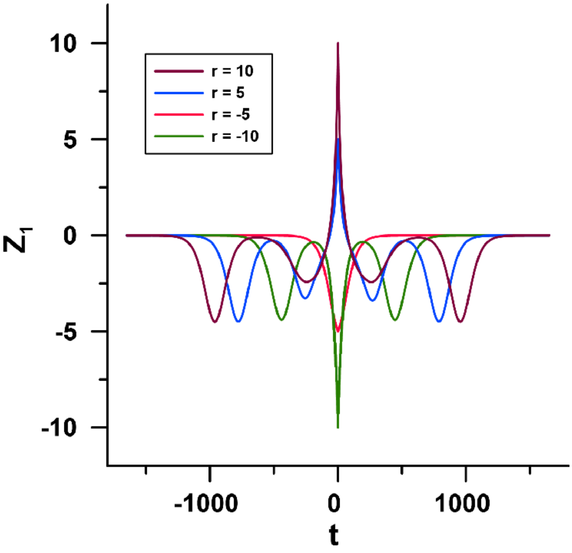


Figure 4: Trajectories of the system (21) at different r

Conclusion

A universal method for studying traveling wave solutions for a wide range of problems in mathematical physics with quasilinear and strongly nonlinear potentials is proposed.

Acknowledgments

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Thank you