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Bistability of globally synchronous and chimera states in a ring of phase oscillators coupled by a cosine kernel

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***Abstract*—Chimera states, where coherent and incoherent activity coexists in homogeneous networks, have been a focus of synchronization theory studies over many years. In this paper, we consider dynamical regimes in a ring of phase oscillators coupled by a cosine kernel using new synchronization criteria - adaptive coherence measure (ACM). We show that the ACM-criterion can be successfully applied for phase oscillator networks. Our measure allowed us to partition the parameter plane into regions along collective dynamics. We discovered that, for the certain parameter sets, there is the bistability between globally synchronous and chimera states. This bistability allows to control network dynamics by changing the initial conditions and/or the external forcing. This shows the potential flexibility in control over complex network behaviors.**

***Index Terms*—chimera state, phase oscillator networks, bistability, adaptive coherence measure, synchronization**

I. Introduction

Chimera states, where coherent and incoherent activity coexist in structurally homogeneous networks, has been of interest to network dynamics and synchronization theory for many years. Initially, such states were discovered in a non-locally coupled complex Ginzburg-Landau equation [1] and with the term ”chimera” appearing in [2] where a ring of phase oscillators symmetrically coupled by a cosine kernel was considered. Later, chimeras were also found in a number of model networks (see, for example, reviews [3, 4]) and observed experimentally [5–10].

There are several methods for identification of network coherent regimes and, in particular, chimera state. The simplest way is to calculate raster plots of network activity, snapshots of instantaneous distributions of the state variable and diagrams of the frequency distribution (e.g. [2, 11]). This approach visually determines the network states for each fixed set of parameters and initial conditions. For a more automatic and generalizable parametric search for chimera states, synchronization measures have been proposed. These are Kuramoto order parameter [2], strength of incoherence [12], and the χ^2-parameter [13]. The *Kuramoto order parameter* is widely used for phase oscillator networks, as well as for neural networks (see, for example, [11, 14, 15]). However, it requires well-defined phases for the network elements. The *strength of incoherence* (in combination with a discontinuity measure) was originally introduced for relaxation systems. It is sensitive to two intrinsic method parameter, the number of bins and the coherence threshold, that must be selected correctly, often separately for each dynamical regime. Such parameter sensitivity renders this method difficulty to apply. A less common χ^2*-parameter* is easier to calculate, yet it has disadvantages that commonly inherent in all these methods. In particular, these methods do not support automatically partitioning of the parameter space of neural networks into areas that correspond to the main dynamic regimes since they confound several of these regimes. There is also a method proposed in [16] that is based on the correlation measure and that allows to distinguish static and travelling chimeras for networks of different nature yet cannot identify comprehensively their coherent states.

To resolve the issues above, we introduce a new robust universal approach that allows to identify *automatically* chimera states [17]. The approach is based on the adaptive coherence measure (ACM) which, in fact, is an generalization of the χ^2-parameter. This approach is free from the above-mentioned drawbacks and can be used to identify correctly and comprehensively dynamical regimes in networks: global and cluster synchronization, chimera states, and travelling waves. We also note that although the approach was originally created for spiking neuronal networks [17], it also works well for various classes of networks and, in particular, for networks of phase oscillators. To verify this, in this work we consider the a ring of phase oscillators symmetrically coupled by a cosine kernel [2] and analyse coherent states of the system.

II. Network description

To show the applicability of our approach for the phase oscillator networks, we consider a Kuramoto ring network [2]:

$$\frac{\partial\phi(x,t)}{\partial t} = \omega - B \sum_{x'=0}^{2\pi} G(x-x')(\sin(\phi(x,t)-\phi(x',t)+\alpha)), \quad (1)$$

Here $x \in [0, 2\pi]$ is a discrete coordinate of oscillators in the ring. The total number of oscillators (N) is equal 256. The coupling $G(x-x')$ between oscillators x and x' is symmetrical and depends on the distance between them:

$$G(x-x') = \frac{1}{2\pi}\left(1 + A\cos(x-x')\right) \qquad (2)$$

Analyzing the dynamical regimes of the network, we choose A and β ($\beta = \pi/2 - \alpha$) as the control parameters and fixed the others ($\omega = 0, B = 1$).

III. Methods

A. Adaptive coherence measure

Let us first describe the adaptive coherence measure. To construct it, we use, as the basis, the χ^2-parameter [13]:

$$\chi^2 = \frac{\sigma_\phi^2}{\frac{1}{N}\sum_{i=1}^{N}\sigma_{\phi_i}^2}, \qquad (3)$$

where $\phi_i = \phi(x_i, t)$ is the phase time series of i-th element in the representative time window and σ_ϕ^2 is variance of average phase of the network $\phi(t) = \frac{1}{N}\sum_{i=1}^{N}\phi(x_i, t)$:

$$\sigma_V^2 = \langle\phi^2(t)\rangle_t - \langle\phi(t)\rangle_t^2, \qquad (4)$$

and $\sigma_{\phi_i}^2$ is variance of the phase of the i-th element:

$$\sigma_{\phi_i}^2 = \langle\phi^2(x_i,t)\rangle_t - \langle\phi(x_i,t)\rangle_t^2. \qquad (5)$$

The χ^2-parameter (as well as Kuramoto order parameter and the strength of incoherence) cannot distinguish travelling waves, cluster synchronization and chimera states. To resolve the problem, we make an advance, that permits us to correctly identify coherent states consisting only of synchronous clusters firing with given time lags. In fact, computing ACM-parameter involves solving the following optimization problem (like it was done for the lag synchronization of two interacting element [18]):

$$R^2 = \max_{\mathbf{\Delta t}=(\Delta t_1, \Delta t_2, .., \Delta t_N)} \chi^2(\{\phi(x_i, t-\Delta t_i)\}_{i=1}^{N}), \qquad (6)$$

As a result, we obtain a vector of time lags $\mathbf{\Delta t} = (\Delta t_1, \Delta t_2, .., \Delta t_N)$ and corresponding values of R^2. A number of unique time lags, let it call L, and the value of R^2 will characterize a dynamical regime (Table I).

Using the ACM-parameter and the number of unique time lags L, one can scan the parameter space and partition it into different dynamical regions:

B. Numerical simulations

For numerical integration of the equations (1) we used the Euler method with a fixed step of 0.025 and total simulation duration of 200 000 iterations.

Table I
TABLE PRESENTING CLASSIFICATION OF THE DIFFERENT NETWORK STATES

Regime	ACM	dimension of $\mathbf{\Delta t}$	number of clusters
Asynchronous state	$R^2 = 0$	$-$	$-$
Global synchronization	$R^2 = 1$	$L = 1$	L
Cluster synchronization	$R^2 = 1$	$1 < L \ll N$	L
Travelling wave	$R^2 = 1$	$L = N$	$-$
Chimera state	$0 < R^2 < 1$	$-$	$-$

IV. Results

Applying the ACM-approach to the system (1), we were able to divide the parameter plane (β, A) into regions with different coherent states. A two-parameter diagram is shown in Fig. 1: the green region corresponds to a chimera state (Fig. 2) and globally synchronous state (Fig. 3) exists in the yellow one. The solid and dashed lines were taken from Fig. 2 in [2]: the solid line corresponds to the boundary determined by numerical solution and the dashed line is approximate boundary obtained from the perturbation theory. One can see that the ACM-approach provides correct results and the boundary of the chimera region plotted using it corresponds well with the numerical boundary calculated in [2], especially for the smaller values of A.

We further calculated the boundary for the globally synchronous region (that was not done in [2]) and found that there exists an intersection between the regions in the Fig. 1. This means that bistability is observed between the chimera and globally synchronous states that co-exist for the same parameter set depending only on the initial conditions. The states corresponding to the intersection region (point "4,5" in the Fig. 1) are demonstrated in the Fig. 4,5. This bistability allows to control the state of the ring network (1) by changing the initial conditions or applying an external forcing.

V. Conclusion

In this paper, we show that the ACM-approach can be successfully used to identify different dynamic regimes and multistability in phase oscillator networks. We would like to note that the classical Kuramoto [2] order parameter has been mainly used for phase oscillator networks, but can also be applied to neuronal networks under certain specific circumstances (as, for example, in [11, 14, 15]). The opposite situation is observed for the ACM parameter: we originally proposed it for spiking neuronal networks, but it also works well for networks of various natures and, in particular, for the networks of phase oscillators. In comparison with the order parameter, the ACM-parameter is able to distinguish *automatically* between chimera states, cluster synchronization, as well as travelling waves (Table I) and has not internal method parameters , which makes the calculations simple and robust.

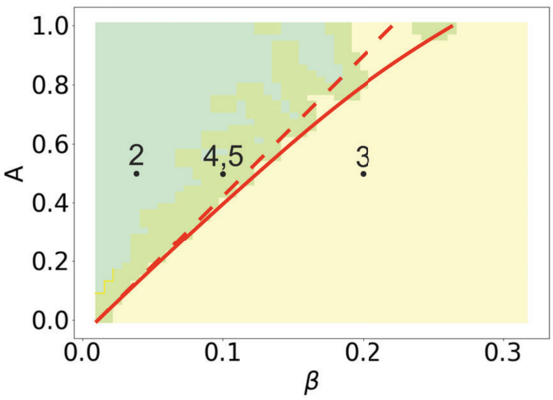

Figure 1. Two parameter (β, A) map for $\omega = 0, B = 1$ presenting regions of dynamical regimes of the ring network (1): chimera state (green), global synchronization (yellow).There is also a bistability region (light green), where, depending on the initial conditions, both dynamic modes can be observed. The solid and dashed lines are taken from Fig. 2 in [2]: the solid line corresponds to the boundary determined by the numerical solution and the dashed line is approximate boundary obtained from the perturbation theory. The points 2,3,4,5 correspond to the numbers of the figures that show the diagrams for each dynamical state of the network (1) for these points.

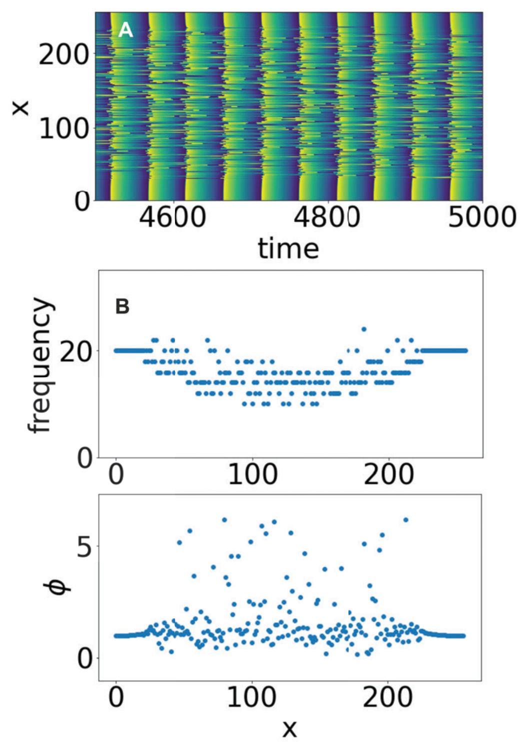

Figure 2. Chimera state corresponding to the point "2" in the Fig. 1 $(A = 0.5, \beta = 0.03)$: rasterplot (A), frequency diagram (B) and instantaneous snapshot (C).

Summing up, we would like to assume that our approach is universal and is able to reliably determine various coherent states of networks, distinguishing, for example, cluster and chimera states, and also, in the case of cluster synchronization, determine the number and size of synchronous clusters. In other words, we recommend the ACM parameter to be used

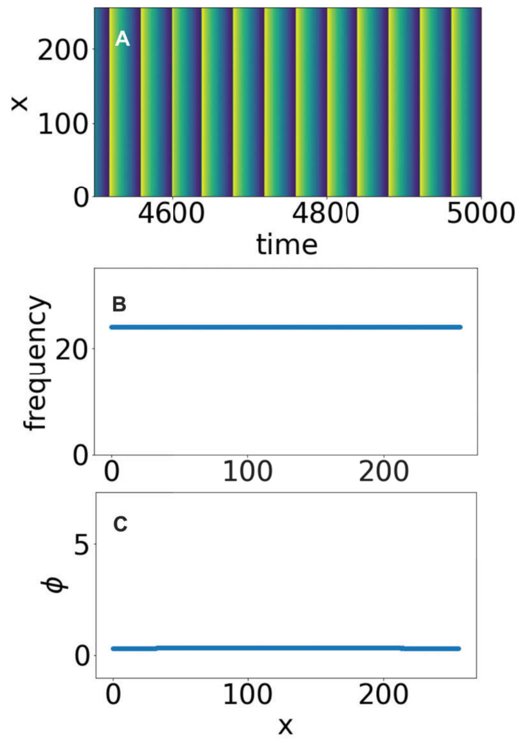

Figure 3. Globally synchronous state corresponding to the point "3" in the Fig. 1 $(A = 0.5, \beta = 0.2)$: rasterplot (A), frequency diagram (B) and instantaneous snapshot (C).

for classification problems of the network dynamical regimes and parameter space partitioning into areas corresponding to these regimes.

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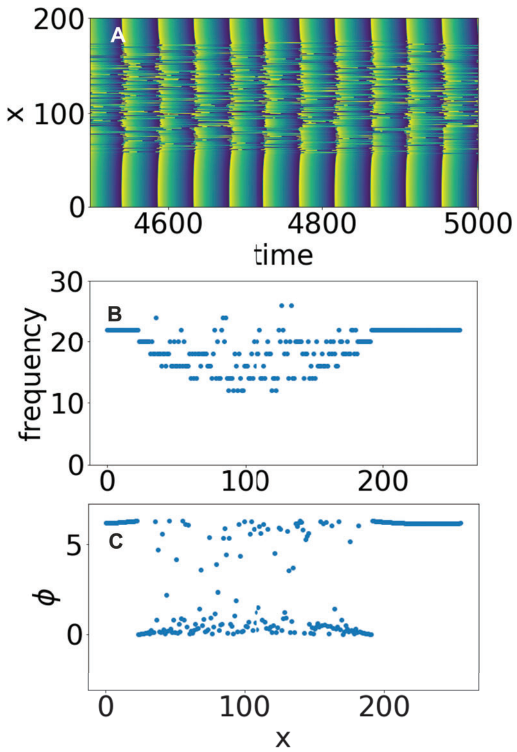

Figure 4. Chimera state in the bistable region (point "4,5" in the Fig. 1) ($A = 0.5, \beta = 0.1$): rasterplot (A), frequency diagram (B) and instantaneous snapshot (C).

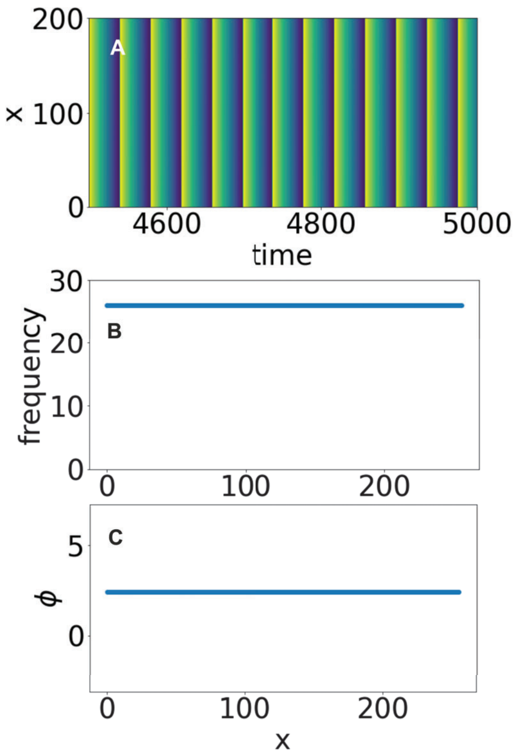

Figure 5. Globally synchronous state in the bistable region (point "4,5" in the Fig. 1) ($A = 0.5, \beta = 0.1$): rasterplot (A), frequency diagram (B) and instantaneous snapshot (C).

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